Full automorphism groups of association schemes based on isotropic subspaces

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Abstract

The set of all subspaces of a given dimension in a finite classical polar space has a structure of a symmetric association scheme. If the dimension is zero, this is the scheme of the collinearity graph of the space; If the dimension is maximum, it is the dual polar scheme. In this note, we determine the full automorphism group of this scheme.

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1 Introduction

A *d-class association scheme* $\mathfrak{X}$ is a pair $(X, \{R_i\}_{i=0}^d)$, where X is a finite set, and each R_i is a nonempty subset of $X \times X$ satisfying the following axioms:

(i) $R_0 = \{(x,x) \,|\, x \in X\}$;

(ii) $X \times X = R_0 \cup R_1 \cup \cdots \cup R_d$, $R_i \cap R_j = \emptyset$ $(i \neq j)$;

(iii) ${}^tR_i = R_{i'}$ for some $i' \in \{0,1,\ldots,d\}$, where ${}^tR_i = \{(y,x) \,|\, (x,y) \in R_i\}$;

(iv) for all $i,j,k \in \{0,1,\ldots,d\}$, there exists an integer

$$p_{ij}^k = |\{z \in X \,|\, (x,z) \in R_i,\, (z,y) \in R_j\}|$$

for every $(x,y) \in R_k$.

The integers p_{ij}^k are called the *intersection numbers* of $\mathfrak{X}$, and $n_i\,(= p_{ii'}^0)$ are called the valencies of $\mathfrak{X}$. Furthermore, $\mathfrak{X}$ is called *symmetric* if $i' = i$ for all i. Let σ be a permutation on X. If σ induces a permutation $\bar{\sigma}$ on $\{0,1,\ldots,d\}$ by $(\sigma(x),\sigma(y)) \in R_{\bar{\sigma}(i)}$ for $(x,y) \in R_i$, then σ is called an *automorphism* of $\mathfrak{X}$. The set of all automorphisms of $\mathfrak{X}$ forms a group,

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called the *full automorphism group* of $\mathfrak{X}$, denoted by $\mathrm{Aut}(\mathfrak{X})$. The readers can consult [1] for the general theory of association schemes.

Let V be an n-dimensional vector space over a finite field $\mathbb{F}_q$. Let also f be a non-degenerate symplectic, hermitian or symmetric form on V of Witt index d. Denote by $\mathcal{N}_m$ be the set of all m-dimensional totally isotropic subspaces and define

$$R_{i,j} = \{(P,Q) \in \mathcal{N}_m \times \mathcal{N}_m | \dim(P^\perp \cap Q) = m - i, \dim(P \cap Q) = m - j\}.$$

Then $\mathfrak{X}_m = (\mathcal{N}_m, \{R_{i,j}\}_{0 \le i \le j \le m})$ is an association scheme. See [5] or [6].

The scheme $\mathfrak{X}_1$ is the scheme of the collinearity graph of the polar space associated to f, and the scheme $\mathfrak{X}_d$ is the dual polar scheme.

In this note, we show that all valencies $n_{i,j}$ of $\mathfrak{X}_m$ are pairwise distinct (Proposition 3). Using this fact, we determine the full automorphism group of $\mathfrak{X}_m$.

Theorem 1 *Let $d \ge 2$ and $1 \le m \le d$. Then*

$$\mathrm{Aut}(\mathfrak{X}_m) = \begin{cases} (\mathrm{Sym}(q+1) \times \mathrm{Sym}(q+1)).2, & \text{if } n = 2d = 4,\ f \text{ is symmetric}, \\ G, & \text{otherwise}, \end{cases}$$

where G is $P\Gamma Sp(V)$, $P\Gamma U(V)$ or $P\Gamma O(V)$ if f is symplectic, hermitian or symmetric, respectively.

2 Valencies of $\mathfrak{X}_m$

In this section we shall prove that all valencies $n_{i,j}$ of the scheme $\mathfrak{X}_m$ are pairwise distinct.

We begin with a useful lemma.

Lemma 2 *Let $f(x)$ be a polynomial of degree at least 1 over the rational number field $\mathbb{Q}$.*

(i) *If $f(x)$ has the following two factorizations:*

$$f(x) = (x^{r_1} - 1)^{i_1}(x^{r_2} - 1)^{i_2} \cdots (x^{r_l} - 1)^{i_l} = (x^{t_1} - 1)^{j_1}(x^{t_2} - 1)^{j_2} \cdots (x^{t_h} - 1)^{j_h},$$

where $r_1 > \cdots > r_l$ and $t_1 > \cdots > t_h$, then $h = l$, $r_s = t_s$ and $i_s = j_s$, $s = 1, \ldots, l$.

(ii) *If $f(x)$ has the following two factorizations:*

$$f(x) = (x^r + 1)(x^{r_1} - 1)^{i_1} \cdots (x^{r_l} - 1)^{i_l} = (x^{t_1} - 1)^{j_1} \cdots (x^{t_h} - 1)^{j_h},$$

then there exists an $s \in \{1, \ldots, h\}$ such that $t_s = 2r$.

(iii) *If $f(x)$ has the following two factorizations:*

$$f(x) = (x^{k_1} + 1) \cdots (x^{k_f} + 1)(x^{r_1} - 1)^{i_1} \cdots (x^{r_l} - 1)^{i_l} = (x^{t_1} - 1)^{j_1} \cdots (x^{t_h} - 1)^{j_h},$$

where $k_1, \ldots, k_f$ are distinct odd integers, then there exist $s_1, \ldots s_f \in \{1, \ldots, h\}$ such that $t_{s_1} = 2k_1, \ldots, t_{s_f} = 2k_f$.

Proof. (i) Consider the term of second minimum degree of $f(x)$, we have

$$(-1)^{i_1 + \cdots + i_l - 1} i_l x^{r_l} = (-1)^{j_1 + \cdots + j_h - 1} j_h x^{t_h},$$

which implies that $j_h = i_l$ and $t_h = r_l$. So

$$(x^{r_1}-1)^{i_1}\cdots(x^{r_{l-1}}-1)^{i_{l-1}} = (x^{t_1}-1)^{j_1}\cdots(x^{t_{h-1}}-1)^{j_{h-1}},$$

By induction, (i) holds.

(ii) The polynomial $(x^r-1)f(x)$ has the following two factorizations:

$$(x^r-1)f(x) = (x^{2r}-1)(x^{r_1}-1)^{i_1}\cdots(x^{r_l}-1)^{i_l} = (x^r-1)(x^{t_1}-1)^{j_1}\cdots(x^{t_h}-1)^{j_h}.$$

So (ii) holds by (i).

The proof of (iii) is similar to that of (ii), and will be omitted. □

Let $\mu = \frac{1}{2}n - d$ and ν be such a number that $\mu+\nu$ equals $0, \frac{1}{2}, 1$ in the symplectic, unitary, orthogonal cases, respectively.

In [6], Wei and Wang computed all valencies $n_{i,j}$ of $\mathfrak{X}_m$ as follows:

$$n_{i,j} = q^{j^2+i(n-2m-2j+\frac{3}{2}i+\frac{1}{2}-\mu-\nu)}\begin{bmatrix} m \\ j \end{bmatrix}\begin{bmatrix} j \\ i \end{bmatrix}\prod_{s=0}^{j-i-1}(q^{\frac{n}{2}-m-\mu-s}-1)(q^{\frac{n}{2}-m-\nu-s}+1)(q^{s+1}-1)^{-1}.$$

Now we show that all $n_{i,j}$ are pairwise distinct.

Proposition 3 *Suppose $1 \le m \le d$. Then all valencies $n_{i,j}$ of $\mathfrak{X}_m$ are pairwise distinct.*

Proof. Suppose $n_{a,b} = n_{a',b'}$. Since q is a prime power, we obtain

$$b^2 + a(n-2m-2b+\frac{3}{2}a+\frac{1}{2}-\mu-\nu) = b'^2 + a'(n-2m-2b'+\frac{3}{2}a'+\frac{1}{2}-\mu-\nu), \qquad (1)$$

and

$$\begin{aligned} & \left[\begin{smallmatrix} m \\ b \end{smallmatrix}\right]\left[\begin{smallmatrix} b \\ a \end{smallmatrix}\right]\prod_{s=0}^{b-a-1}(q^{\frac{n}{2}-m-\mu-s}-1)(q^{\frac{n}{2}-m-\nu-s}+1)(q^{s+1}-1)^{-1} \\ = & \left[\begin{smallmatrix} m \\ b' \end{smallmatrix}\right]\left[\begin{smallmatrix} b' \\ a' \end{smallmatrix}\right]\prod_{s=0}^{b'-a'-1}(q^{\frac{n}{2}-m-\mu-s}-1)(q^{\frac{n}{2}-m-\nu-s}+1)(q^{s+1}-1)^{-1}. \end{aligned} \qquad (2)$$

Simplifying (1), we have

$$(b-b')(2b+2b'-4a') = (a'-a)(2l+3a'+3a-4b), \qquad (3)$$

where $l = n-2m+\frac{1}{2}-\mu-\nu$. Write

$$\begin{aligned} f_{i,j}(x) &= \prod_{s=0}^{j-i-1}(x^{n-2m-2\mu-2s}-1)(x^{n-2m-2\nu-2s}+1), \\ g_{i,j}(x) &= \prod_{s=1}^{m-j}(x^{2s}-1)\prod_{s=1}^{i}(x^{2s}-1)\prod_{s=1}^{j-i}(x^{2s}-1)^2. \end{aligned}$$

The equality (2) implies that $f_{a,b}(q)g_{a',b'}(q) = f_{a',b'}(q)g_{a,b}(q)$ for all prime powers q, so

$$f_{a,b}(x)g_{a',b'}(x) = f_{a',b'}(x)g_{a,b}(x). \qquad (4)$$

Computing the degree of this polynomial, we have

$$(b-b')(2l-5b-5b'+m+8a') = (a'-a)(-2l-5a-5a'+8b). \qquad (5)$$

Equalities (3) and (5) imply that

$$(b-b')(2l+2m-b-b') = (a'-a)(2l+a+a'). \qquad (6)$$

Suppose $b \neq b'$. Without loss of generality, we assume that $b > b'$. Since $2l+2m-b-b' > 0$ and $2l + a + a' > 0$, by (6) one gets $a' > a$ and

$$b - a - b' + a' \geq 2. \tag{7}$$

Write

$$\begin{aligned} k(x) &= \prod_{s=b'-a'+1}^{b-a} (x^{2s} - 1)^2, \\ h(x) &= \prod_{s=b'-a'}^{b-a-1} (x^{n-2m-2\mu-2s} - 1)(x^{n-2m-2\nu-2s} + 1) \prod_{s=m-b+1}^{m-b'} (x^{2s} - 1) \prod_{s=a+1}^{a'} (x^{2s} - 1). \end{aligned}$$

The equation (4) implies that $k(x) = h(x)$.

Write $h_1(x) = x^{2n-4m-4\nu-4b'+4a'} - 1$ and $h_2(x) = x^{2n-4m-4\nu-4b+4a+4} - 1$. By Lemma 2 there exist $s_1, s_2 \in \{b' - a' + 1, \ldots, b - a\}$ such that $h_1(x) = x^{2s_1} - 1$ and $h_2(x) = x^{2s_2} - 1$. It follows that $s_1 = n - 2m - 2\nu - 2b' + 2a'$ and $s_2 = n - 2m - 2\nu - 2b + 2a + 2$. By (7), we have $s_1 - s_2 = 2b - 2a - 2b' + 2a' - 2 > b - a - (b' - a' + 1)$, a contradiction. So $b = b'$ and $a = a'$, as desired. $\square$

3 Proof of Theorem 1

By Proposition 3, any automorphism σ of $\mathfrak{X}_m$ preserves every $R_{i,j}$. In particular, σ is an automorphism of the polar Grassmann graph $(\mathcal{N}_m, R_{0,1})$.

It is well-known that the full automorphism group of $(\mathcal{N}_1, R_{0,1})$ is isomorphic to $(\mathrm{Sym}(q+1) \times \mathrm{Sym}(q+1)).2$ or G according to $n = 2d = 4$ and f is symmetric or not, we refer [3] for the details. By [2] every automorphism of $(\mathcal{N}_d, R_{0,1})$ is induced by an automorphism of $(\mathcal{N}_1, R_{0,1})$. Hence Theorem 1 holds for the two cases $m = 1$ and $m = d$.

Now we consider the case $1 < m < d$. It follows from [4, Theorem 4.8] that all automorphisms of $\mathfrak{X}_m$ are induced by automorphisms of $(\mathcal{N}_1, R_{0,1})$ except the following special case: $n = 2d = 8, m = 2$ and f is symmetric.

From this moment we suppose that $n = 2d = 8, m = 2$ and f is symmetric. In this case, $\mathcal{N}_4$ is the disjoint union of the half-spin Grassmannians $\mathcal{N}_+$ and $\mathcal{N}_-$, see [3] or [4, Subsection 4.2.3]. One of the remarkable properties of half-spin Grassmannians is the following: if $P, Q \in \mathcal{N}_\delta$, $\delta \in \{+, -\}$, then $d - \dim(P \cap Q)$ is even; if $P \in \mathcal{N}_+$ and $Q \in \mathcal{N}_-$ then this number is odd.

Consider the following relation $R_{\delta,1}$ on $\mathcal{N}_\delta$, $\delta \in \{+, -\}$:

$$(P, Q) \in R_{\delta,1} \ \text{ if } \ \dim(P \cap Q) = 2.$$

The half-spin Grassmannian $\mathcal{N}_\delta$ admits the natural structure of a polar space and $(\mathcal{N}_\delta, R_{\delta,1})$ is the collinearity graph of this polar space. The graphs $(\mathcal{N}_\delta, R_{\delta,1})$ and $(\mathcal{N}_1, R_{0,1})$ are isomorphic. Every isomorphism of $(\mathcal{N}_1, R_{0,1})$ to $(\mathcal{N}_\delta, R_{\delta,1})$ induces an automorphism of $(\mathcal{N}_2, R_{0,1})$ [4, p. 160]. By [4, Theorem 4.9], every automorphism of $\mathfrak{X}_2$ is induced by an automorphism of $(\mathcal{N}_1, R_{0,1})$ or an isomorphism of $(\mathcal{N}_1, R_{0,1})$ to $(\mathcal{N}_\delta, R_{\delta,1})$.

Let σ be the second type automorphism of $\mathfrak{X}_2$. Suppose $(X, Y) \in R_{0,2}$. Then $U := X + Y \in \mathcal{N}_4$. By [4, Subsection 4.6.5], we have $(\sigma(X), \sigma(Y)) \in R_{1,1}$ if $U \in \mathcal{N}_\delta$ and $(\sigma(X), \sigma(Y)) \in R_{0,2}$ otherwise, a contradiction. Hence, any automorphism of $\mathfrak{X}_2$ is induced by an automorphism of $(\mathcal{N}_1, R_{0,1})$.

By above discussion, the proof of Theorem 1 is completed. $\square$

4 Final remarks

Suppose that our form f is symmetric and $n = 2d \geq 4$. By [4, Subsection 4.2.3], $\mathcal{N}_d$ is the disjoint union of the half-spin Grassmannians $\mathcal{N}_\delta$, $\delta \in \{+, -\}$. If $P, Q \in \mathcal{N}_\delta$, then $d - \dim(P \cap Q)$ is even, if $P \in \mathcal{N}_+$ and $Q \in \mathcal{N}_-$, then this number is odd. Let

$$R_{\delta,2i} = \{(P, Q) \in \mathcal{N}_\delta \times \mathcal{N}_\delta \mid \dim(P \cap Q) = d - 2i\}.$$

Then $\mathfrak{X}_{\delta,d} := (\mathcal{N}_\delta, \{R_{\delta,2i}\}_{0 \leq 2i \leq d})$ is a subscheme of the dual polar scheme $\mathfrak{X}_d$. The valency of $R_{\delta,2i}$ is just the valency $n_{2i,2i}$ of $\mathfrak{X}_d$.

Note that $(\mathcal{N}_\delta, R_{\delta,2})$ is a halved graph of the dual polar graph $(\mathcal{N}_d, R_{1,1})$. If $d \leq 3$, this graph is a complete graph. If $d \geq 4$ then the graph has precisely two types of maximal cliques [4, Subsection 4.5.2]; cliques of different types have the same number of vertices only in the case when $d = 4$.

By [2] and [3], the full automorphism group of $(\mathcal{N}_\delta, R_{\delta,2})$ is $\mathrm{P\Gamma O}^+(V, f)$ if $d \geq 5$.

In the case when $d = 4$, every automorphism of $(\mathcal{N}_\delta, R_{\delta,2})$ is induced by an element of $\mathrm{P\Gamma O}^+(V, f)$ or an isomorphism of $(\mathcal{N}_1, R_{0,1})$ to $(\mathcal{N}_{-\delta}, R_{\delta,2})$ [4, Theorem 4.10]. The automorphisms of the second type can be characterized as automorphisms of $(\mathcal{N}_\delta, R_{\delta,2})$ which change the type of every maximal clique. Thus $\mathrm{P\Gamma O}^+(V, f)$ is a subgroup of index 2 in the full automorphism group of $(\mathcal{N}_\delta, R_{\delta,2})$ which implies that the the full automorphism group of $(\mathcal{N}_\delta, R_{\delta,2})$ is isomorphic to $\mathrm{P\Gamma O}(V, f)$.

By Proposition 3, every automorphism of $\mathfrak{X}_{\delta,d}$ is an automorphism of $(\mathcal{N}_\delta, R_{\delta,2})$. Hence the full automorphism group of the scheme $\mathfrak{X}_{\delta,d}$ is isomorphic to $\mathrm{P\Gamma O}(V, f)$ or $\mathrm{P\Gamma O}^+(V, f)$ according to $d = 4$ or $d \geq 5$.

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